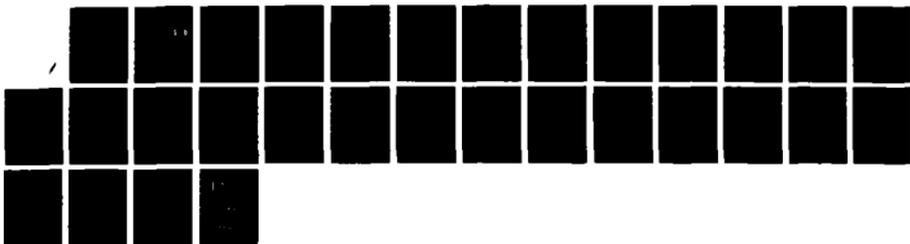


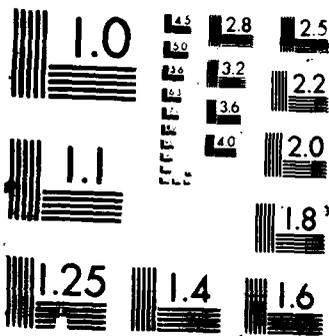
AD-A187 435 LOCAL BIFURCATION CONTROL(U) MARYLAND UNIV COLLEGE PARK 1/1
E H ABED 1987 AFOSR-TR-87-1443 AFOSR-87-8873

UNCLASSIFIED

F/G 25/5

NL





AD-A187 435

AFOSR-TR. 87-1443

1

To appear in "Dynamical Systems Approaches to Nonlinear Problems in Systems and Circuits", M. Levi and F.M.A. Salam, Eds., Siam Publ., Philadelphia, 1987

DTIC FILE COPY

Local Bifurcation Control

EYAD H. ABED · 1987

DTIC
ELECTE
NOV 16 1987
S H D

Abstract. Local feedback stabilization of bifurcated solution branches is studied. Two cases are considered: that in which the nominal system undergoes a Hopf bifurcation as a parameter is varied, and the case of a stationary bifurcation from a simple zero eigenvalue. For each case, results on the existence of a stabilizing feedback are given. Moreover, simple synthesis techniques for the stabilizing controllers are discussed. A concept of "proximity stabilization" is introduced as an alternative to stabilization in the ordinary sense for systems that are not locally stabilizable. A result is stated on the genericity of proximity stabilizability. Motivation for further research in several areas is given.

1. Introduction

A standard preliminary step in the analysis and design of control systems is the linearization of the model dynamics about a nominal reference trajectory. The analysis and/or design are then performed for the resulting approximate linear model. The success of this technique in many applications can be attributed to the result of Liapunov that, if the linearized system is locally asymptotically stable, so too is the original nonlinear model. This is the celebrated "principle of linearized stability." This principle holds for finite systems of ordinary differential equations, as well as for some infinite dimensional problems. Situations in which the local stability properties of the nonlinear model cannot be inferred from its linearization are referred to as *critical cases* in stability theory [15].

* Department of Electrical Engineering and Systems Research Center, University of Maryland, College Park, MD 20742 USA. Research supported in part by the National Science Foundation under Grants ECS-84-04275 and CDR-85-00108, and by AFOSR-URI Grant No AFOSR-87-0073

DISTRIBUTION STATEMENT A

Approved for public release;
Distribution Unlimited

87 10 14 343

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION		1b. RESTRICTIVE MARKINGS		
2a. SECURITY CLASSIFICATION AUTHORITY		3. DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release ; distribution unlimited.		
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE				
4. PERFORMING ORGANIZATION REPORT NUMBER(S)		5. MONITORING ORGANIZATION REPORT NUMBER(S) AFO SR - TR - 87 - 1443		
6a. NAME OF PERFORMING ORGANIZATION	6b. OFFICE SYMBOL <i>(If applicable)</i>	7a. NAME OF MONITORING ORGANIZATION		
6c. ADDRESS <i>(City, State, and ZIP Code)</i>		7b. ADDRESS <i>(City, State, and ZIP Code)</i>		
8a. NAME OF FUNDING / SPONSORING ORGANIZATION	8b. OFFICE SYMBOL <i>(If applicable)</i>	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFO SR - 87 - 0013		
8c. ADDRESS <i>(City, State, and ZIP Code)</i>		10. SOURCE OF FUNDING NUMBERS		
		PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.
11. TITLE <i>(Include Security Classification)</i>				
12. PERSONAL AUTHOR(S)				
13a. TYPE OF REPORT	13b. TIME COVERED FROM _____ TO _____	14. DATE OF REPORT <i>(Year, Month, Day)</i> 1987	15. PAGE COUNT	
16. SUPPLEMENTARY NOTATION				
17. SOCAT CODES		18. SUBJECT TERMS <i>(Continue on reverse if necessary and identify by block number)</i>		
FIELD	GROUP			SUB-GROUP
19. ABSTRACT <i>(Continue on reverse if necessary and identify by block number)</i>				
20. DISTRIBUTION / AVAILABILITY OF ABSTRACT <input type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT <input type="checkbox"/> DTIC USERS		21. ABSTRACT SECURITY CLASSIFICATION		
22a. NAME OF RESPONSIBLE INDIVIDUAL		22b. TELEPHONE <i>(Include Area Code)</i>	22c. OFFICE SYMBOL	

Early results on stability in critical cases were obtained by Liapunov [25] , Malkin [26] , Pliss [33] and Krasovskii [23] . This topic has continued to be an active area of investigation. Recently, the related question of stabilization in critical cases has received significant attention. The papers [12, 8, 35, 4, 1, 2, 16, 7] represent a broad spectrum of approaches to this problem.

Our aim in this lecture is to summarize some recent results of the author and J.-H. Fu [1, 2] on the closely related problem of local *bifurcation control*, to present an extension to one case where the results of [1, 2] do not apply, and to indicate some directions for further work in this area. By "local bifurcation control" we mean the local stabilization of bifurcated solutions. Two types of bifurcation from an equilibrium point of a one-parameter family of ordinary differential equations are considered. First, the standard Hopf bifurcation of a periodic solution; and second, stationary (or static) bifurcation, involving only equilibrium points. The connection between local stabilization in critical cases and local bifurcation control becomes transparent given some basic facts about bifurcations of equilibria of differential equations. There are advantages to analyzing these two problems simultaneously. Often results on stabilization in critical cases can be directly applied also to problems in the control of bifurcations. Correspondingly, local bifurcation control problems provide added motivation for the study of stabilization in critical cases.

Our approach to the local feedback stabilization problem has the novel feature that it facilitates the derivation of generally valid analytical criteria for stabilizability, as well as specific stabilizing feedback controls. This is possible through use of bifurcation formulae which involve only Taylor series expansion of the vector field and eigenvector computations. The bifurcation formulae used in [1] for the study of control of Hopf bifurcations were obtained by Howard [19] by a harmonic balance approach using the Fredholm Alternative. These formulae significantly simplify similar formulae obtained by Hopf [18] . In our study of stationary bifurcations, we derive bifurcation formulae following the Projection Method, as outlined in looss and Joseph [21] . This is essentially the same technique used by Howard [19] in his study of Hopf bifurcations.

Previous work on the feedback control of bifurcations includes that of Mehra [28] and Mehra, Kessel and Carroll [29] . See also the account in Casti [10] . These studies tend to be concerned with the problem of globally *removing* bifurcations by state feedback. The general results apply only to stationary bifurcations, since they are obtained by appealing to a global implicit function theorem. This is in contrast to the *local* bifurcation control problems considered here, where one seeks only to modify the stability properties of the bifurcated solutions.

The development of the paper is as follows. The problems of local feedback stabilization and local bifurcation control are discussed in Section 2. In Section 3, the main results on control of Hopf bifurcations are recalled from [1] without proof. In Section 4 known results on stationary bifurcations are recalled, and bifurcation formulae are derived for this case. These results are applied to the stationary bifurcation control problem in Section 5, summarizing the main results of [2] . A notion of "proximity stabilization" is introduced in Section 6, and shown to be useful in problems for which no stabilizing control law exists in the usual sense. Section 7 contains two worked



For	<input checked="" type="checkbox"/>
AI	<input type="checkbox"/>
ed	<input type="checkbox"/>
tion	<input type="checkbox"/>
on/	
ality Codes	
Avail and/or	
Special	
Dist	
A-1	

examples, one illustrating the results and the other indicating an area for further investigation related to the proximity stabilization issue of Section 6.

2. Problem Setting

Consider a one-parameter family of nonlinear control systems

$$\dot{x} = f_{\mu}(x, u) \quad (1)$$

where $x \in R^n$, u is a scalar control, μ is a real-valued parameter, and the vector field f_{μ} is sufficiently smooth. Suppose that for $u \equiv 0$ Eq. (1) has an equilibrium point $x_0(\mu)$ which depends smoothly on μ . In the sequel the system

$$\dot{x} = f_0(x, u), \quad (2)$$

which is simply (1) with $\mu = 0$, will also be of interest.

This paper is concerned with the synthesis of feedback controls $u = u(x)$ achieving certain stability properties for each of the descriptions (1) and (2).

The linearization of Eq. (2) at $x = 0$, $u = 0$ is given by

$$\dot{x} = Ax + bu \quad (3)$$

where $A := \frac{\partial f}{\partial x}(0,0)$ and $b := \frac{\partial f}{\partial u}(0,0)$. If the pair (A, b) is controllable,¹ then a standard linear systems result [22] asserts the existence of a linear feedback $u = -kx$ such that the resulting system $\dot{x} = (A - bk)x$ is asymptotically stable. Applying this feedback in the original nonlinear system (2) renders the origin locally asymptotically stable. Moreover, the same conclusion applies if the uncontrollable modes of (3) are asymptotically stable. In contrast, if (A, b) has an unstable uncontrollable mode, then the origin of (2) remains unstable regardless of the applied feedback. These considerations imply that the only essentially nonlinear situation encountered in local feedback stabilization occurs when some uncontrollable modes of (3) are pure imaginary, and any other uncontrollable modes are asymptotically stable. However, even if there are modes which are pure imaginary and controllable, it may be important to study the existence of *nonlinear* stabilizing feedback control laws [34, 31, 32].

It is natural to question the utility of results giving nonlinear stabilizing feedback controls when a linear stabilizing control law exists. There are indeed several reasons for using nonlinear feedbacks, two of which are noted next. First, the effect of a linear feedback control designed to stabilize the linearized version of the critical system (2) on the one-parameter family of systems (1) may be difficult to determine. Indeed, at least for small feedback gains, one can expect that the bifurcation will reappear at a different value of the parameter μ . The stability of this new bifurcation is not easily determined. Hence, simply using a linear stabilizing feedback may be unacceptable if the goal is to stabilize a bifurcation and not merely to stabilize an equilibrium point for a fixed

¹This means that the state x of Eq. (3) can be "steered" from any initial condition x_0 to any prescribed terminal condition x_1 within a prespecified time, by appropriate choice of the control $u(t)$. Similarly, one speaks of controllability of modes of Eq. (3) (eigenvalues of A), depending on whether or not these modes are affected by state feedback.

parameter value. Second, it should not be surprising that in some situations a linear feedback which locally stabilizes an equilibrium may result in globally unbounded behavior, whereas nonlinear feedbacks exist which stabilize the equilibrium both locally and globally. For an example, see Moon and Rand [32]. Hence, even if stabilization, rather than bifurcation control, is the issue being studied, nonlinear feedback controls can be superior.

In the light of the foregoing discussion, it is appropriate to assume that the matrix A of Eq. (3) possesses at least one eigenvalue with zero real part. The type of results obtained will depend heavily on the number of eigenvalues of A which are assumed pure imaginary, their multiplicity, and whether they are zero or have nonzero imaginary parts. The results of this paper address situations in which either of the following two hypotheses is satisfied. The first implies that Eq. (1) undergoes a Hopf bifurcation to periodic solutions at $\mu = 0$ when $u \equiv 0$, while the second ensures that new stationary solutions of (1) bifurcate from $x_0(\mu)$ at $\mu = 0$ when $u \equiv 0$.

(H) Eq. (1) has an equilibrium $x_0(\mu)$ when $u = 0$. Furthermore, the linearization of (1) near x_0 , $\mu = 0$ possesses a pair of simple, complex conjugate eigenvalues $\lambda_1(\mu)$, $\lambda_2(\mu) = \overline{\lambda_1(\mu)}$, with $\lambda_1(0) = i\omega_c$, $\omega_c > 0$, $\text{Re } \lambda_j'(0) \neq 0$, with the remaining eigenvalues $\lambda_3(0), \dots, \lambda_n(0)$ in the open left half complex plane.

(S) Eq. (1) has an equilibrium $x_0(\mu)$ when $u = 0$. Furthermore, the linearization of (1) near x_0 , $\mu = 0$ possesses a simple eigenvalue $\lambda_1(\mu)$ with $\lambda_1(0) = 0$, $\lambda_1'(0) \neq 0$, with the remaining eigenvalues $\lambda_2(0), \dots, \lambda_n(0)$ in the open left half complex plane.

The assumption that $\lambda_j'(0) \neq 0$ is the familiar strict-crossing (transversality) condition introduced by Hopf [18].

Two stabilization problems are considered in the sequel. Both are studied separately for hypotheses (H) and (S). In each case, one of these problems pertains to Eq. (1) and the other to Eq. (2). For Eq. (1), the goal is to ensure local asymptotic stability of the bifurcated solutions. This will be referred to as the *local Hopf bifurcation control problem* or the *local stationary bifurcation control problem*, depending on which hypothesis is in force. For the description (2), it is desired to solve the standard *local feedback stabilization problem* at the equilibrium point $x_0(0)$. Note that under either hypothesis (H) or (S), Eq. (2) with $u \equiv 0$ is an example of a critical nonlinear system since its linearization possesses an eigenvalue with zero real part.

3. Control of Hopf bifurcations

Under hypothesis (H), the Hopf Bifurcation Theorem asserts the existence of a one-parameter family $\{p_\epsilon, 0 < \epsilon \leq \epsilon_0\}$ of nonconstant periodic solutions of Eq. (1) emerging from $x = 0$ at $\mu = 0$. (This assumes $u \equiv 0$, of course.) Here ϵ is a measure of the amplitude of the periodic solutions and ϵ_0 is sufficiently small. The periodic solutions $p_\epsilon(t)$ have period near $2\pi\omega_c^{-1}$ and occur for parameter values μ given by a smooth function $\mu(\epsilon)$. Exactly one of the characteristic exponents of p_ϵ is near 0, and it is given by a real, smooth and even function

$$\beta(\epsilon) = \beta_2 \epsilon^2 + \beta_4 \epsilon^4 + \dots \quad (4)$$

Moreover, $p_\epsilon(t)$ is orbitally asymptotically stable with asymptotic phase if $\beta(\epsilon) < 0$ but is unstable if $\beta(\epsilon) > 0$. Denote by β_{2K} the first nonvanishing coefficient in the expansion (4). Checking the sign of β_{2K} is sufficient for determining stability. Generically, $K = 1$ so that locally the stability of the bifurcated periodic solutions p_ϵ is typically decided by the sign of the coefficient β_2 .

An algorithm for the computation of β_2 can be useful in the solution of local feedback stabilization problems under hypothesis (H). In [4] the evaluation of (a scaled version of) β_2 is performed using a formula which applies to two-dimensional systems. The original n -dimensional system is reduced to a two-dimensional system by appealing to the Center Manifold Theorem. Use is then made of the fact [9, 14, 27, 11] that the stability properties of an equilibrium on the center manifold coincide with its stability in R^n . In fact, the value of $\beta(\epsilon)$ is known [27, 17, 3] to be the same for the original and the reduced systems. The approach taken in this work (cf. Abed and Fu [1]) differs from that of [4] mainly in the choice of algorithm for computing β_2 . The implications for the type of results one obtains are nontrivial.

Now suppose $\beta_2 \neq 0$. Besides locally determining the stability of the bifurcated periodic solutions $p_\epsilon(t)$, it is known that the sign of the coefficient β_2 also *determines the stability of the equilibrium $x_0(\mu)$ at criticality* (i.e. at $\mu = 0$). This fact implies that if a feedback control $u = u(x)$ can be found such that $\beta_2 < 0$ for the Hopf bifurcation occurring in the controlled system

$$\dot{x} = f_\mu(x, u(x)) \quad (5)$$

then the local feedback stabilization problem described in Section 2 is solved. Simply use the feedback $u = u(x)$ in Eq. (1). Indeed, such a feedback solves *both* the local smooth feedback stabilization problem for Eq. (1) *and* the local Hopf bifurcation stabilization problem for *any* parametrized version of (1) of the form (5). This establishes the connection between local feedback stabilization and Hopf bifurcation control.

Rewrite Eq. (1) in the series form

$$\begin{aligned} \dot{x} = & L_0 x + u \gamma + u L_1 x + Q_0(x, x) \\ & + C_0(x, x, x) + \dots \end{aligned} \quad (6)$$

where the terms not written explicitly are of higher order in x , u and μ than those which are. Thus L_0 and L_1 are square matrices, γ is a constant vector, $Q_0(x, x)$ is a quadratic form generated by a symmetric bilinear form $Q_0(x, y)$ giving the second order (in x) terms at $u = 0$, $\mu = 0$, and $C_0(x, x, x)$ is a cubic form generated by a symmetric trilinear form $C_0(x, y, z)$ giving the third order (in x) terms at $u = 0$, $\mu = 0$. (Note that L_0 is simply A of Eq. (3), and γ corresponds to b .)

Denote by r the right (column) and by l the left (row) eigenvector of L_0 with eigenvalue $i\omega_c$. Normalize by setting the first component of r to 1 and then choose l so that $lr = 1$.

It is well known that only the quadratic and cubic terms occurring in a nonlinear system undergoing a Hopf bifurcation influence the value of β_2 . Thus only the linear, quadratic and cubic terms in an applied feedback $u(x)$ have potential for influencing β_2 . To simplify the analysis and to emphasize the influence of nonlinear terms in the feedback control, we require $u(x)$ to be of the form

$$u(x) = x^T Q_u x + C_u(x, x, x), \quad (7)$$

where Q_u is a real symmetric $n \times n$ matrix, and C_u is a cubic form generated by a scalar valued symmetric trilinear form.

Theorems 1 and 2 below give sufficient conditions for local stabilizability of a Hopf bifurcation by feedback of the form (7). Both are positive results. Theorem 1 applies under the hypothesis that $l\gamma \neq 0$, while in Theorem 2 one $l\gamma = 0$ is assumed. By the well known Popov-Belevitch-Hautus (PBH) eigenvector test for controllability [22] of modes of linear time-invariant systems, the former case corresponds to the critical modes being controllable for the linearized system (3), while in the latter case these modes are uncontrollable. The theorems were proved in [1] using the formula for β_2 derived by Howard [19]. Reference [1] also contains specific formulae for stabilizing feedback controls.

Theorem 1. *Let hypothesis (H) hold and assume that $l\gamma \neq 0$. That is, the critical eigenvalues are controllable for the linearized system. Then there is a smooth feedback $u(x)$ with $u(0) = 0$ which solves the local Hopf bifurcation control problem for Eq. (1) and the local feedback stabilization problem for Eq. (2). Moreover, this can be accomplished with only third order terms in $u(x)$, leaving the critical eigenvalues unaffected.*

Theorem 2. *Suppose that hypothesis (H) is satisfied and that $l\gamma = 0$. Then there is a smooth feedback $u(x)$ with $u(0) = 0$ which solves the local Hopf bifurcation control problem for Eq. (1) and the local feedback stabilization problem for Eq. (2) provided that*

$$\begin{aligned} 0 \neq \operatorname{Re} \{ & -2lQ_0(r, \frac{1}{2}L_0^{-1}\gamma) \\ & + lQ_0(r, \frac{1}{2}(2i\omega_c I - L_0)^{-1}\gamma) \\ & + \frac{1}{4}l[2L_1 r + L_1 \bar{r}] \}. \end{aligned} \quad (8)$$

4. Stationary bifurcations: Analysis

Under the stationary bifurcation hypothesis (S), it is well known [19] that Eq. (1) exhibits a stationary (or static) bifurcation from x_0 at $\mu = 0$. That is, new stationary solutions (i.e. equilibrium points) bifurcate from x_0 at $\mu = 0$. The stability characteristics of the new solutions are intimately related to those of $x_0(\mu)$ at criticality, i.e. at $\mu = 0$. It is this intrinsic relationship that allows the joint consideration of local stabilization for Eq. (2) and bifurcation control for Eq. (1).

To establish this relationship and motivate the derivations to follow, consider a general one-parameter family of nonlinear ordinary differential equations

$$\dot{x} = F_{\mu}(x) \quad (9)$$

having an equilibrium point $x_0(\mu)$ at which hypothesis (S) holds. Then near $x_0(0)$, $\mu=0$ in (x, μ) space there exists a locally unique curve of points $(x(\epsilon), \mu(\epsilon))$, distinct from the μ axis and passing through $(0,0)$, such that for all sufficiently small $|\epsilon|$, $x(\epsilon)$ is an equilibrium point of (9) when $\mu = \mu(\epsilon)$. Moreover, the parameter ϵ may be chosen so that $x(\epsilon)$ and $\mu(\epsilon)$ are smooth.

Denote the series expansions of $\mu(\epsilon)$, $x(\epsilon)$ by

$$\mu(\epsilon) = \mu_1\epsilon + \mu_2\epsilon^2 + \dots, \quad (10)$$

$$x(\epsilon) = x_1\epsilon + x_2\epsilon^2 + \dots \quad (11)$$

respectively. Generically, $\mu_1 \neq 0$, and there is a second equilibrium point besides $x_0(\mu)$ for all small $|\mu|$. However, if $\mu_1 = 0$ and $\mu_2 > 0$ (resp. $\mu_2 < 0$), then there are two new equilibrium points, one for positive and negative values of ϵ . These occur only for sufficiently small positive (resp. negative) values of μ . The new equilibrium points also have an eigenvalue β which vanishes at $\mu = 0$, with a series expansion

$$\beta(\epsilon) = \beta_1\epsilon + \beta_2\epsilon^2 + \dots \quad (12)$$

Moreover, the exchange of stability formula [19, 18]

$$\beta_1 = -\mu_1\lambda'_j(0) \quad (13)$$

holds. If $\mu_1 = 0$ and $\mu_2 \neq 0$, the appropriate exchange of stability formula is [18]

$$\beta_2 = -2\mu_2\lambda'_j(0). \quad (14)$$

(Note: Eqs. (13) and (14) may be derived using the Factorization Theorem in Iooss and Joseph [15, pp. 90-91].) Suppose $\lambda'_j(0) > 0$. Then these facts imply that supercritical solution branches are stable while subcritical branches are unstable.

The following result follows from an application of the Center Manifold Theorem to a suspended version of Eq. (9) at $x_0(0)$, $\mu = 0$.

Theorem 3. *Let hypothesis (S) hold. If $\mu_1 \neq 0$, then the equilibrium point $x_0(0)$ is unstable for Eq. (9). If $\mu_1 = 0$ and $\mu_2 \neq 0$, then $x_0(0)$ is asymptotically stable if $\beta_2 < 0$ but is unstable if $\beta_2 > 0$.*

Thus, the equilibrium point $x_0(0)$ will be assured asymptotically stable if one can arrange that $\beta_1 = 0$ and $\beta_2 < 0$. If explicit formulae can be derived for β_1 and β_2 , this provides a starting point for the construction of locally stabilizing feedback controls for the critical system (2). In fact, by the exchange of stability formulae, it is clear that this also ensures the stability of the bifurcated stationary solution, by ensuring that the bifurcation is a supercritical *pitchfork bifurcation*. This is a desirable outcome, as compared to the *transcritical bifurcation* which would occur if $\beta_1 \neq 0$, in which the bifurcated equilibrium point is stable on one side of $\mu = 0$ and unstable on the other. Indeed, under hypothesis (S), a supercritical pitchfork bifurcation ensures that, even though the nominal equilibrium solution $x_0(\mu)$ loses stability as μ varies through 0, the new equilibrium solution attracts a neighborhood of initial conditions about $x_0(0)$. This can also be shown through an application of the Center Manifold Theorem and the

theory of normal forms (see [14] for details).

Next, bifurcation formulae for Eq. (9) are derived. The results will be the main tool in the construction of stabilizing feedbacks for Eqs. (1) and (2) in Section 5. The Projection Method, as elaborated in [21], will be employed in the derivation.

By assumption, the Jacobian matrix $D_x F_0(x_0(0))$ of (9) at criticality possesses a simple zero eigenvalue $\lambda_1(0)$. Denote by r (resp. l) the right column (resp. left row) eigenvector of the critical Jacobian matrix corresponding to this eigenvalue. Using the fact that 0 is a simple eigenvalue, it is not difficult to see that the vectors l and r may be chosen to have only real elements. To be more specific, set the first component of r to 1 and then choose l so that $lr = 1$.

Without loss of generality, assume that for small $|\mu|$ the known equilibrium point $x_0(\mu)$ of (9) is the origin, i.e. $x_0(\mu) \equiv 0$ for small $|\mu|$. This can always be achieved by a smooth change of variables $x \rightarrow x + x_0(\mu)$. Rewrite (9) in the series form

$$\begin{aligned} \dot{x} &= L(\mu)x + Q_\mu(x, x) + C_\mu(x, x, x) + \dots \\ &= L_0x + \mu L_1x + \mu^2 L_2x + \dots \\ &\quad + Q_0(x, x) + \mu Q_1(x, x) + \dots \\ &\quad + C_0(x, x, x) + \dots \end{aligned} \quad (15)$$

Here, $L(\mu)$, L_1 , L_2 are $n \times n$ matrices, $Q_\mu(x, x)$, $Q_0(x, x)$, $Q_1(x, x)$ are vector valued quadratic forms generated by symmetric bilinear forms $Q_\mu(x, y)$, $Q_0(x, y)$, $Q_1(x, y)$, respectively, and $C_0(x, x, x)$ is a vector valued cubic form generated by a symmetric trilinear form $C(x, y, z)$. The terms not explicitly written in (15) are of higher order in x and μ than those which are.

A convenient outcome of this representation is the formula

$$\lambda_1'(0) = ll_1r. \quad (16)$$

See [19] or [21] for a proof.

If x is any real (unknown) solution of $F_\mu(x) = 0$, define the parameter ϵ by

$$\epsilon := lx, \quad (17)$$

and attempt a series expansion of the form

$$\begin{pmatrix} x(\epsilon) \\ \mu(\epsilon) \end{pmatrix} = \sum_{k=1}^{\infty} \epsilon^k \begin{pmatrix} x_k \\ \mu_k \end{pmatrix}. \quad (18)$$

Substituting the expansion (18) in the equation obtained by equating the right side of (15) to 0, and equating coefficients of like powers of ϵ yields the following relationships.

$$0 = L_0x_1, \quad (19)$$

$$0 = L_0x_2 + \mu_1 L_1x_1 + Q_0(x_1, x_1), \quad (20)$$

$$0 = L_0 x_3 + \mu_1 L_1 x_2 + \mu_2 L_1 x_1 + \mu_1^2 L_2 x_1 + 2Q_0(x_1, x_2) + \mu_1 Q_1(x_1, x_1) + C_0(x_1, x_1, x_1). \quad (21)$$

By Eqs. (17) and (18),

$$\begin{aligned} \epsilon &= lx(\epsilon) \\ &= \epsilon lx_1 + \epsilon^2 lx_2 + \epsilon^3 lx_3 + \dots \end{aligned} \quad (22)$$

Hence,

$$lx_1 = 1, \text{ and } lx_k = 0 \text{ for } k \geq 2. \quad (23)$$

Eqs. (19) and (23), and the assumption that 0 is a simple eigenvalue of L_0 , now imply

$$x_1 = r. \quad (24)$$

Substituting this in Eq. (20) gives the following equation, which should evidently be solved for *both* x_2 and μ_1 :

$$L_0 x_2 = -\mu_1 L_1 r - Q_0(r, r). \quad (25)$$

Recall that L_0 is singular. From elementary linear algebra (or the Fredholm Alternative), this equation has a solution x_2 if and only if the right side of (25) is orthogonal to all left eigenvectors of L_0 corresponding to the zero eigenvalue. Since zero is a simple eigenvalue of L_0 , one need only require that

$$\mu_1 lL_1 r + lQ_0(r, r) = 0, \quad (26)$$

so that μ_1 is determined as

$$\mu_1 = -\frac{1}{\lambda'_j(0)} lQ_0(r, r) \quad (27)$$

where Eq. (16) has been employed.

Since the Fredholm Alternative conditions are now satisfied, Eqs. (20) and (23) for x_2 have a solution. This solution is easily verified to be unique. Equations (20), (23) are conveniently expressed as the single equation

$$\begin{pmatrix} L_0 \\ l \end{pmatrix} x_2 = \begin{pmatrix} -\mu_1 L_1 r - Q_0(r, r) \\ 0 \end{pmatrix}. \quad (28)$$

Since (28) has a unique solution, the coefficient matrix

$$R := \begin{pmatrix} L_0 \\ l \end{pmatrix} \quad (29)$$

is full rank. Hence, $R^T R$ is a nonsingular square matrix and x_2 is given by

$$x_2 = (R^T R)^{-1} R^T \begin{pmatrix} -\mu_1 L_1 r - Q_0(r, r) \\ 0 \end{pmatrix}. \quad (30)$$

With x_2 now available, one applies the Fredholm Alternative to Eq (21) to solve

for the coefficient μ_2 . Multiplying both sides of (21) by l and solving for μ_2 , one obtains

$$\begin{aligned} \mu_2 = -\frac{1}{\lambda_1'(0)} \{ & \mu_1 l L_1 x_2 + \mu_1^2 l L_2 r + 2l Q_0(r, x_2) \\ & + \mu_1 l Q_1(r, r) + l C_0(r, r, r) \}. \end{aligned} \quad (31)$$

Using the exchange of stability formula (13) and Eq. (27) for μ_1 , the coefficient β_1 is found to be

$$\beta_1 = l Q_0(r, r). \quad (32)$$

If $\mu_1 = 0$ (implying also $\beta_1 = 0$), then the exchange of stability formula (14) is valid. In that case, one finds that β_2 is given by

$$\beta_2 = 2l (2Q_0(r, x_2) + C_0(r, r, r)), \quad (\text{if } \mu_1 = 0 \text{ only}). \quad (33)$$

The formulae (27), (30)-(33) will be employed in the next section to obtain sufficient conditions for bifurcation controllability and local stabilizability for Eqs. (1) and (2), respectively.

5. Stationary bifurcations: Control

Motivated by the bifurcation formulae derived above, and by the results of Section 3, one expands the vector field of Eq. (1) as

$$\begin{aligned} \dot{x} &= f_\mu(x, u) \\ &= L_0 x + \mu L_1 x + u \bar{L}_1 x + u \gamma + Q_0(x, x) \\ &\quad + \mu^2 L_2 x + \mu Q_1(x, x) + u \bar{Q}_1(x, x) \\ &\quad + C_0(x, x, x) + \dots \end{aligned} \quad (34)$$

The notation here is similar to that in Eq. (15). As in Section 3, a feedback control consisting of quadratic and cubic terms is assumed. That is, $u = u(x)$ is taken as

$$u(x) = x^T Q_u x + C_u(x, x, x), \quad (35)$$

where Q_u is a real symmetric $n \times n$ matrix and $C_u(x, x, x)$ is a cubic form generated by a scalar valued symmetric trilinear form. Note that, as in Section 3, $u(x)$ contains no terms linear in x . This ensures that the left and right eigenvectors corresponding to the zero eigenvalue, and the value of μ at criticality, will be unaffected by the feedback control. The closed loop dynamics with a feedback of the form (35) become (starred quantities below denote values after feedback)

$$\begin{aligned} \dot{x} &= L_0^* x + Q_0^*(x, x) + C_0^*(x, x, x) \\ &\quad + \mu L_1^* x + \mu^2 L_2^* x + \mu Q_1^*(x, x) + \dots \end{aligned} \quad (36)$$

where the matrices L_i^* , $i = 0, 1, 2$, the quadratic forms $Q_0^*(x, x)$, $Q_1^*(x, x)$ and the cubic form $C_0^*(x, x, x)$ are

$$L_i^{\circ} = L_i, \quad i = 0, 1, 2, \quad (37a)$$

$$Q_0^{\circ}(x, x) = (x^T Q_u x) \gamma + Q_0(x, x) \quad (37b)$$

$$Q_i^{\circ}(x, x) = Q_i(x, x) \quad (37c)$$

and

$$C_0^{\circ}(x, x, x) = C_u(x, x, x) \gamma + C_0(x, x, x) + (x^T Q_u x) \bar{L}_1 x. \quad (37d)$$

Symmetric bilinear and trilinear forms $Q_0^{\circ}(x, y)$, $C_0^{\circ}(x, y, z)$ generating the quadratic and cubic forms $Q_0^{\circ}(x, x)$ and $C_0^{\circ}(x, x, x)$, respectively, are now chosen:

$$Q_0^{\circ}(x, y) = (x^T Q_u y) \gamma + Q_0(x, y). \quad (38)$$

$$C_0^{\circ}(x, y, z) = C_u(x, y, z) \gamma + C_0(x, y, z) + \frac{1}{3} \{ (y^T Q_u z) \bar{L}_1 x + (x^T Q_u y) \bar{L}_1 z + (z^T Q_u x) \bar{L}_1 y \}. \quad (39)$$

After feedback, the coefficient β_1 becomes, using Eq. (32),

$$\begin{aligned} \beta_1^{\circ} &= l Q_0^{\circ}(r, r) \\ &= l \{ Q_0(r, r) + (r^T Q_u r) \gamma \} \\ &= \beta_1 + (r^T Q_u r) l \gamma, \end{aligned} \quad (40)$$

where β_1 denotes the value of β_1 with no feedback, i.e. with $u(x) \equiv 0$. From (40) it is clear that a sufficient condition for the existence of a feedback $u(x)$ driving β_1 to 0 is

$$l \gamma \neq 0. \quad (41)$$

Indeed, if (41) holds, then any feedback control of the form (35) with

$$r^T Q_u r = -\frac{\beta_1}{l \gamma} \quad (42)$$

results in $\beta_1^{\circ} = 0$.

Recall from the Popov-Belevitch-Hautus (PBH) eigenvector test for controllability of modes of linear time-invariant systems [22] that $l \gamma \neq 0$ is equivalent to controllability of the zero eigenvalue of the linearized system corresponding to Eq. (2) near the origin. Thus, controllability of the critical zero eigenvalue for the linearized system is sufficient for the existence of a feedback ensuring $\beta_1^{\circ} = 0$. Indeed, if $\beta_1 \neq 0$, this condition is also necessary, as can be seen from Eq. (40).

As outlined in Section 4, the next step after ensuring that $\beta_1^{\circ} = 0$ is to arrange, if possible, that $\beta_2^{\circ} < 0$. By Eq. (33), β_2° is given by

$$\beta_2^{\circ} = 2l (2Q_0^{\circ}(r, x_2^{\circ}) + C_0^{\circ}(r, r, r)). \quad (43)$$

To proceed, it is necessary to evaluate x_2° , according to the formula (30) derived in Section 4. Since $u(x)$ contains no linear terms, the matrix R occurring in Eq. (30) is the

same before and after feedback, as is clear from Eq. (29). The fact that $\mu_j^* = 0$ also simplifies the expression for x_2^* . One has

$$\begin{aligned} x_2^* &= -(R^T R)^{-1} R^T \begin{pmatrix} Q_0^*(r, r) \\ 0 \end{pmatrix} \\ &= -(R^T R)^{-1} R^T \begin{pmatrix} Q_0(r, r) + (r^T Q_u r) \gamma \\ 0 \end{pmatrix} \\ &= x_2 - (r^T Q_u r) (R^T R)^{-1} R^T \begin{pmatrix} \gamma \\ 0 \end{pmatrix}. \end{aligned} \quad (44)$$

From (44), it is clear that only the quadratic terms (i.e., Q_u) in the feedback control influence the value of x_2^* . Using (38) and (39), one evaluates β_2^* :

$$\begin{aligned} \beta_2^* &= 2l \{ 2Q_0(r, x_2^*) + 2(r^T Q_u x_2^*) \gamma + (r^T Q_u r) \bar{L}_1 r \\ &\quad + C_0(r, r, r) + C_u(r, r, r) \gamma \} \\ &= 2l \gamma C_u(r, r, r) + \phi, \end{aligned} \quad (45)$$

where ϕ is a quantity which is fixed once Q_u is chosen, and, moreover, does not depend on the trilinear form $C_u(x, y, z)$.

From Eq. (45) it follows that if $l\gamma \neq 0$, the value of β_2^* may be assigned arbitrarily by appropriate choice of the scalar valued trilinear form $C_u(x, y, z)$. This holds regardless of the choice made for the symmetric matrix Q_u . The following theorem summarizes these results.

Theorem 4. *Let hypothesis (S) hold and assume $l\gamma \neq 0$, that is, the critical zero eigenvalue is controllable for the linearized version of Eq. (2) near the origin. Then there is a smooth feedback control $u = u(x)$ with $u(0) = 0$, containing only quadratic and cubic terms in x , which solves the local stationary bifurcation control problem for Eq. (1) and the local smooth feedback stabilization problem for Eq. (2). Moreover, the quadratic terms in $u(x)$ can be used to ensure that $\beta_1 = 0$ for the controlled system, and the cubic terms can then be used to ensure that $\beta_2 < 0$.*

Theorem 4 should be compared with Theorem 1, which contains the analogous results for local Hopf bifurcation control. The assumption $l\gamma \neq 0$ was sufficient for stabilizability in that setting as well. However, a stabilizing feedback consisting of only cubic terms was needed, while both quadratic and cubic terms are required in Theorem 4. This is due to the need for a two-stage control design in the stationary bifurcation control case.

Now consider the case $l\gamma = 0$, i.e. let the critical (zero) eigenvalue be uncontrollable for the linearized system. In the setting of Section 3, under the analogous assumption it was found that generically local feedback stabilization of the nonlinear system is achievable. However, Eq. (40) reveals that in the present setting feedback has no effect on the value of β_1 in case $l\gamma = 0$. The discussion in Section 4 therefore implies that the local feedback stabilization problem for Eq. (2) will then be unsolvable, unless

perhaps it happens that $\beta_1 = 0$ in the absence of a control effort (a nongeneric assumption). Similarly, the local stationary bifurcation control problem is also generically unsolvable in case $l\gamma = 0$.

Theorem 5. *Let hypothesis (S) hold and assume $l\gamma = 0$, that is, the critical zero eigenvalue is uncontrollable for the linearized version of (2). Then if $\beta_1 \neq 0$ for Eq. (1) with $u(x) \equiv 0$, both the local stationary bifurcation control problem for Eq. (1) and the local feedback stabilization problem for Eq. (2) are not solvable by a smooth feedback control with vanishing linear part.*

Note that the negative conclusion of this theorem does not exclude the possibility that "nearly stabilizing" feedback controls might be constructed for the case $l\gamma = 0$. In the next section we consider this point, and introduce precise notions of "proximity stabilization" and "proximity stabilizability" as an alternative to local stabilization for this case.

6. Proximity stabilization

Under the assumptions that a simple zero eigenvalue is linearly uncontrollable and $\beta_1 \neq 0$, Theorem 5 above asserts that smooth local feedback stabilization is unattainable within the class of purely nonlinear feedback controls. This fact may lead one to search for other acceptable forms of local stability besides the standard notion of asymptotic stability in the sense of Liapunov. In this context, recall that Brockett [8] has obtained easily verifiable necessary conditions for stabilizability by any smooth feedback control. Consider the following definition.

Definition 1. The origin is said to be *proximity stabilizable* for Eq. (2) if for any $\epsilon > 0$ there is a smooth feedback $u = u(x)$ rendering the ball centered at the origin of radius ϵ in R^n a locally attracting set.

To illustrate the nature of proximity stabilization, we consider a simple scalar example. Suppose x satisfies

$$\dot{x} = x^2 - ux. \quad (46)$$

Then $\beta_1 \neq 0$ and $l\gamma = 0$, so there does not exist a smooth feedback $u(x)$ containing no linear terms stabilizing the origin. Indeed, it is easily checked that no smooth feedback can render the origin asymptotically stable, even with linear terms in x . However, this system is proximity stabilizable in the sense of Definition 1. This is easily seen by noting that $\{ |x| < \epsilon \}$ is attracting for (46) if $u(x) = kx^2$ with $k > \epsilon^{-1}$.

The following result states that under the assumptions of Theorem 5, generically there is a smooth feedback control containing only cubic terms in the state x solving the proximity stabilization problem. Our proof is not difficult but in the interest of brevity it will be presented elsewhere.

Theorem 6. *Under the hypotheses of Theorem 5, the proximity stabilization problem for Eq. (2) is solvable with a cubic feedback control under generic assumptions on the function f .*

7. Examples and directions for further research

The first example considered below was studied previously by Su, Meyer and Hunt [35]. They derived a locally stabilizing feedback control law using feedback linearization as an intermediate step. In principle, this technique allows one to construct an infinite family of stabilizing controllers for the example. This example is one of a nonlinear critical system, and does not involve a bifurcation parameter. The second example is that of a parametrized scalar equation, and is studied mainly to motivate consideration of an issue of proximity stabilization (in a sense to be discussed below) of bifurcations.

Example 1. Following Su, Meyer and Hunt [35], consider the control system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \sin x_2 \\ 0 \end{pmatrix} + v \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (47)$$

where v is the control. Note that the linearization of (47) possesses a double zero eigenvalue. Thus the theory of Section 5 cannot be applied directly. To circumvent this problem, we define a new control u by

$$u = v + \alpha x_2$$

where α is an arbitrary positive number. Then (47) becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \sin x_2 \\ -\alpha x_2 \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (48)$$

A suitably parametrized version of (48) would, if $u = 0$, undergo a stationary bifurcation from a simple zero eigenvalue. Thus the results of Section 5 are applicable to Eq. (48). Note that no quadratic terms appear in the series expansion of the right side of (48) for $u = 0$. Hence $Q_0(x, y) \equiv 0$. By (27), this implies $\mu_1 = 0$, and hence also $\beta_1 = 0$ for the uncontrolled system. Theorem 4 then implies that, if $l\gamma \neq 0$, only cubic terms would be needed to stabilize the origin of Eq. (48). Indeed, it is easily seen that $r = (1 \ 0)^T$, $l = (1 \ \alpha^{-1})$, and since $\gamma = (0 \ 1)^T$,

$$l\gamma = \alpha^{-1} \neq 0. \quad (49)$$

The coefficient β_2 for the uncontrolled system can be found using Eq. (33). We have

$$\begin{aligned} \beta_2 &= 2l C_0(r, r, r) \\ &= 2(1 \ \alpha^{-1})(0) \\ &= 0. \end{aligned} \quad (50)$$

Hence stability cannot be determined based only on knowledge of β_1 and β_2 . However, using Eq. (45) and the preceding comments, we find that the value of β_2 after control is applied is

$$\beta_2^* = 2\alpha^{-1} C_u(r, r, r) \quad (51)$$

and we are free to choose the cubic terms C_u in the feedback control. (Recall that the quadratic terms Q_u have been set to 0.) It is easy to check that the choice

$$u = C_u(x, x, x) = -0.5 \alpha \delta x_1^3, \quad (52)$$

$\delta > 0$, renders $\beta_2' < 0$. Hence, for the original system (47), we have the stabilizing control laws

$$v = -\alpha x_2 - \delta x_1^3 \quad (53)$$

for any $\alpha, \delta > 0$. The essential distinction between this control law and that obtained in [35] is that linear feedback in only one variable appears in our control law, while linear feedback in both variables forms part of the control obtained in [35]. Computations involving the nonlinear terms are essential in deriving the control law (53) above.

Example 2 (Motivational). Consider the parametrized scalar equation

$$\dot{x} = \mu x - \alpha x^3 + 2\alpha^2 x^9 \quad (54)$$

where μ is the bifurcation parameter and α is a design parameter. We wish to view (54) as arising from a specific choice of the form of a feedback controller in a parametrized scalar control system. The term $2\alpha^2 x^9$ has been retained since we are now mainly interested in more global considerations. It is easy to check that any choice of $\alpha > 0$ will stabilize the (stationary) bifurcation occurring at $\mu = 0, x = 0$.

Suppose, however, that we are also interested in ensuring the persistence of some sort of local asymptotic stability for a wide range of values of the parameter $\mu > 0$. Since the origin is unstable for $\mu > 0$, we will be satisfied with rendering a neighborhood of the origin attracting. If no further restrictions are placed on the problem, the solution is quite simple: given an interval $0 \leq \mu \leq \mu_0$ of parameter values for which attractivity of a neighborhood of the origin is desired, there is an α sufficiently large achieving this goal. This follows by a simple Liapunov function argument, viewing (54) as a perturbed version of the stable equation

$$\dot{z} = -\alpha z^3. \quad (55)$$

The difficulty, of course, is that the size of the neighborhood which is attracting may shrink drastically for large μ . In a practical sense, therefore, this approach may prove unacceptable.

Consider the following alternative formulation: Given a "stability region tolerance" $\epsilon_0 > 0$, find $\alpha > 0$ to maximize the value μ_1 such that the set $|x| < \epsilon_0$ is locally attracting for $0 \leq \mu \leq \mu_1$. The following analysis addresses this issue. Before proceeding, it should be noted that the important goal of maximizing the size of the region of attraction of the ϵ_0 neighborhood is also an important goal, but is not incorporated into this problem formulation.

Define the Liapunov function candidate

$$V(x) = x^2/2. \quad (56)$$

The derivative of V along trajectories of (54) is

$$\dot{V}(x) = x^2 (\mu - \alpha x^2 + 2\alpha^2 x^8). \quad (57)$$

Now $\dot{V} < 0$ at $x = \pm\epsilon_0$ if and only if

$$\mu - \alpha \epsilon_0^2 + 2 \alpha^2 \epsilon_0^8 < 0, \quad (58)$$

i.e., if and only if

$$\mu < \mu_{\max}(\alpha) := \alpha \epsilon_0^2 (1 - 2 \alpha \epsilon_0^6). \quad (59)$$

Now

$$\max_{\alpha > 0} \mu_{\max}(\alpha) = \mu_{\max}(\alpha^*), \quad (60)$$

where

$$\alpha^* = 0.25 \epsilon_0^{-8} \quad (61)$$

is the best value of α that could be obtained with the Liapunov function V . Incidentally, μ_1 will then be given by

$$\mu_1 = \epsilon_0^{-4}/8. \quad (62)$$

This simple example is given as motivation for studying more general versions of this proximity stabilization issue for bifurcations. The basic issues noted for Example 2 recur in the analysis of the general case. The stability of critical systems depending on parameters has received much attention in the past. Parameter values for which the origin is stable for the critical system are termed *safe* while those for which the origin is unstable are known as *dangerous* [15, 5]. This terminology arose from the consideration of the stability properties of maneuvering military aircraft. This application is still an active area of investigation [29, 30, 20]. The author is continuing these efforts by investigating application of the results discussed here to the aircraft high angle of attack flight control problem.

References

- [1] E.H. ABED and J.-H. FU, "Local feedback stabilization and bifurcation control, I. Hopf bifurcation," *Systems and Control Letters* **7**, pp. 11-17, 1986.
- [2] E.H. ABED and J.-H. FU, "Local feedback stabilization and bifurcation control, II. Stationary bifurcation," *Systems and Control Letters* **8**, 1987, to appear.
- [3] E.H. ABED, "A simple proof of stability on the center manifold for Hopf bifurcation," *SIAM Review* **29**, 1987, to appear.
- [4] D. AEYELS, "Stabilization of a class of nonlinear systems by a smooth feedback control," *Systems and Control Letters* **5**, pp. 289-294, 1985.
- [5] J.R. AMYOT and R.A. JOHNSON, "On two theories concerning the dangerousness of stability boundaries in nonlinear systems," pp. 748-754 in *Proc. Joint Automatic Control Conf.*, Troy, N.Y., 1965.
- [6] V.I. ARNOLD, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York, 1983.

- [7] S. BEHTASH and S.S. SASTRY, "Stabilization of nonlinear systems with degenerate linearization," pp. 912-915 in *Proc. 1986 IEEE Int. Symp. Circ. Syst.*, San Jose, May 1986.
- [8] R.W. BROCKETT, "Asymptotic stability and feedback stabilization," pp. 181-191 in *Differential Geometric Control Theory*, ed. R.W. BROCKETT, R.S. MILLMAN and H.J. SUSSMANN, Birkhauser, Boston, 1983.
- [9] J. CARR, *Applications of Centre Manifold Theory*, Springer-Verlag, New York, 1981.
- [10] J. CASTI, *Connectivity, Complexity, and Catastrophe in Large-Scale Systems*, Wiley-Interscience, Chichester, 1979.
- [11] S.-N. CHOW and J.K. HALE, *Methods of Bifurcation Theory*, Springer-Verlag, New York, 1982.
- [12] A. KH. GELIG and O.I. KOMARNITSKAYA, "Absolute stability in critical cases of nonlinear systems with a nonunique equilibrium state," *Automation and Remote Control* 27, 8, pp. 1327-1337, 1966.
- [13] J. Guckenheimer, "Persistent properties of bifurcations," *Physica* 7D, pp. 105-110, 1983.
- [14] J. GUCKENHEIMER and P. HOLMES, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.
- [15] W. HAHN, *Stability of Motion*, Springer-Verlag, New York, 1967.
- [16] V.T. HAIMO, "An algebraic approach to nonlinear stability," *Nonlinear Analysis, Th. Meth. Appl.* 10, pp. 683-692, 1986.
- [17] B.D. HASSARD, N.D. KAZARINOFF, and Y.-H. WAN, *Theory and Applications of Hopf Bifurcation*, Cambridge Univ. Press, Cambridge, 1981.
- [18] E. HOPF, "Bifurcation of a periodic solution from a stationary solution of a system of differential equations," *Ber. Math.-Phys. Kl. Sachs. Akad. Wiss. Leipzig* XCIV, pp. 1-22 (Translation to English with commentary in [29]), 1942.
- [19] L.N. HOWARD, "Nonlinear oscillations," pp. 1-68 in *Nonlinear Oscillations in Biology*, ed. F.C. Hoppensteadt, Amer. Math. Soc., Providence, R.I., 1979.
- [20] W.H. HUI and M. TOBAK, "Bifurcation analysis of aircraft pitching motions about large mean angles of attack," *J. Guidance* 7, pp. 113-122, 1984.

- [21] G. IOOSS and D.D. JOSEPH, *Elementary Stability and Bifurcation Theory*, Springer-Verlag, New York, 1980.
- [22] T. KAILATH, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.
- [23] N.N. KRASOVSKII, *Stability of Motion: Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay*, Stanford Univ. Press, Stanford, CA, 1963.
- [24] M. KUBICEK and M. MAREK, *Computational Methods in Bifurcation Theory and Dissipative Structures*, Springer-Verlag, New York, 1983.
- [25] A.M. LIAPUNOV, *Stability of Motion*, Academic Press, New York, 1966.
- [26] I. MALKIN, "Some basic theorems of the stability of motion in the critical cases," *AMS Translations* 38, 1958.
- [27] J.E. MARSDEN and M. McCracken, *The Hopf Bifurcation and Its Applications*, Springer-Verlag, New York, 1976.
- [28] R.K. MEHRA, "Catastrophe theory, nonlinear system identification and bifurcation control," pp. 823-831 in *Proc. of the Joint Automatic Control Conference, 1977*.
- [29] R.K. MEHRA, W.C. KESSEL, and J.V. CARROLL, "Global Stability and Control Analysis of Aircraft at High Angles-of-Attack," ONRCR-215-248-1, U.S. Office of Naval Research, Arlington, VA, June 1977.
- [30] R.K. MEHRA and J.V. CARROLL, "Bifurcation analysis of aircraft high angle-of-attack flight dynamics," pp. 127-146 in *New Approaches to Nonlinear Problems in Dynamics*, ed. P.J. Holmes, SIAM, Philadelphia, 1980.
- [31] G.I. MELNIKOV, "Angular stabilization of an aircraft vehicle by means of nonlinear control," *Automation and Remote Control* 27, 8, pp. 1338-1345, 1966.
- [32] F.C. MOON and R.H. RAND, "Parametric stiffness control of flexible structures," pp. 329-338 in *NASA Conf. Proc.: Workshop on Identification and Control of Flexible Space Structures*, San Diego, May 1984.
- [33] V.A. PLISS, "An investigation of a transcendental case of the theory of stability," pp. 185-201 in *Stability of Motion (by A.M. LIAPUNOV)*, Academic Press, New York, 1966.
- [34] E.P. POPOV, "Nonlinear control laws in automation," *Izv. AN SSSR, Energetika i avtomatika* 5, 1965.
- [35] R. SU, G. MEYER, and L.R. HUNT, "Robustness in nonlinear control," pp. 316-337 in *Differential Geometric Control Theory*, ed. R.W. Brockett, R.S. Millman and H.J. Sussmann, Birkhauser, Boston, 1983.

Appendix 3

P. S. KRISHNAPRASAD

Associate Professor of Electrical Engineering

Dr. Krishnaprasad received his Ph.D degree from Harvard University in 1977. He was on the faculty of the Systems Engineering Department at Case Western Reserve University from 1977 till 1980. He has been with the University of Maryland since August 1980. At Maryland he is a member of the Faculty of the Applied Mathematics Program and the Systems Research Center. He has held visiting positions with Erasmus University (Rotterdam) and with the Department of Mathematics, University of California, Berkeley. Dr. Krishnaprasad has participated in consulting activities with the Army Night Vision and Electro Optics Laboratory at Ft. Belvoir.

Dr. Krishnaprasad has directed 3 Ph.D students and 5 M.S. students. He is currently supervising 3 Ph.D students and 6 M.S. students and several undergraduate researchers in a variety of projects.

Dr. Krishnaprasad was a member of the Stability, Nonlinear and Distributed Parameter Systems Committee of the IEEE Transactions on Automatic Control. He is now on the Editorial Board of Systems and Control Letters. He is a reviewer for IEEE, the National Science Foundation, SIAM Journals and others. His professional society memberships include the IEEE, the AIAA and the AMS.

Dr. Krishnaprasad has contributed to the growth of basic research in control theory by organizing special sessions at a variety of conferences on topics related geometric control theory and dynamical systems.

RESEARCH INTERESTS

Dr. Krishnaprasad's research interests lie in the broad area of geometric control theory and its applications. He has contributed to the understanding of parametrization problems in linear systems, the Lie algebraic foundations of certain nonlinear filtering problems pertaining to system identification, the Lie theory and stability of interconnected mechanical systems (e.g., space craft with elastic attachments and spinning rotors) and symmetry principles in nonlinear control theory.

Dr. Krishnaprasad's current research interests include mathematical problems in the kinematics of robot manipulators, the real-time control of a flexible robot arm with end-point sensing, and the development of symbolic algebraic tools for analysis and design in robotics. These activities have led to the creation of the Intelligent Control Systems Laboratory and a new package(DYNMAN) in MACSYMA for design and analysis of robotic manipulators.

SELECTED RECENT PAPERS

- (1) "Lie-Poisson Structures, Dual-Spin Spacecraft and Asymptotic Stability" (1985) *Nonlinear Analysis: Theory, Methods and Applications*, vol. 9, No. 10, 1011-1035.
- (2) "The Self-Tuning Controller: Comparison with Human Performance in the Control of Arterial Pressure" (1985) *Annals of Biomedical Engineering*, vol. 13, 341-357. (with K.Stern, B.Walker, H.Chizeck, M.Dauchot, P.Katona).
- (3) "Hamiltonian Structures and Stability for Rigid Bodies with Flexible Attachments: Part I" (1985) accepted for publication in *Archive for Rational Mechanics and Analysis*, with (J. E. Marsden); also *Report of the Center for Pure and Applied Mathematics*, University of California, Berkeley, PAM-276, 38 pages.
- (4) "A Cell Decomposition for Rational Functions" (1985) to appear in *IMA Journal of Mathematical Control and Information*, special issue on *Parametrization Problems*, (with P.

A. Fuhrmann, invited) also Technical Research Report, Systems Research Center, University of Maryland, TR-85-32, 28 pages.

- (5) "DYNAMAN: A Tool for Manipulator Design and Analysis" (1985) , to appear in the proceedings of the *IEEE International Conference on Robotics and Automation, April 7 - 10, 1986, San Francisco*, also Technical Research Report TR-85-12, Systems Research Center, University of Maryland, 39 pages.
- (6) "Control of Interconnected Mechanical Systems" invited paper to be presented at the *Symposium on Temporal and Spatial Chaos, 10th U. S. National Congress of Applied Mechanics, Austin, Texas, June 1986*.

EYAD H. ABED

Education

Ph.D. in Electrical Engineering, University of California at Berkeley,
December 1982. Dissertation Title: *Singularly Perturbed Hopf Bifurcation
and Oscillations in Power Systems*. Research Advisor: Pravin P. Varaiya.

M.S. in Electrical Engineering, University of California at Berkeley,
December 1981.

S.B. in Electrical Engineering, Massachusetts Institute of Technology,
Cambridge, June 1979.

Present Position

Assistant Professor, Electrical Engineering Department,
University of Maryland, College Park.

Professional Awards

George Corcoran Award for Significant Contribution to Electrical Engineering
Education, College of Engineering, University of Maryland, College Park, 1983.

Outstanding Professor Award, IEEE Student Branch, University of Maryland,
College Park, 1983.

Selected Recent Publications

"Nonlinear oscillations in power systems," *Int. J. Electrical
Power and Energy Systems*, Vol. 6, No. 1, pp. 37-43, Jan. 1984.

"Singularly perturbed Hopf bifurcation," *IEEE Trans. Circuits
and Systems*, Vol. CAS-32, No. 12, pp. 1270-1280, December 1985.

"Multiparameter singular perturbation problems: Iterative expansions
and asymptotic stability," *Systems and Control Letters*, Vol. 5,
No. 4, pp. 279-282, February 1985.

"Decomposition and stability of multiparameter singular perturbation
problems," *IEEE Trans. Automatic Control*, to appear, 1986.

(with J.-H. Fu), "Local feedback stabilization and bifurcation control,
I. Hopf bifurcation," *Systems and Control Letters*, to appear, 1986.

"A simple proof of stability on the center manifold for Hopf
bifurcation," *SIAM Review*, to appear, 1986.

(with A.L. Tits), "On the stability of multiple time-scale systems,"
Int. J. Control, to appear, 1986.

STUART S. ANTMAN

Professor of Mathematics

Stuart Antman was born in Brooklyn, New York on June 2, 1939. He received his B.S. from the Rensselaer Polytechnic Institute in 1961. He received both the M.S. and Ph.D. degrees from the University of Minnesota in 1963 and 1965 respectively.

Before coming to the University of Maryland in 1972, Dr. Antman was Assistant and Associate Professor at the New York University from 1967-1972. He has held numerous visiting professor positions internationally. Dr. Antman is a member of AMS, MAA, Pi Mu epsilon, SIAM, Society of Natural Philosophy and the American Academy of Mechanics. Additionally, he participates in many activities of service to the Academic Community.

RESEARCH INTERESTS

Professor Antman's main research interests are in the areas of Nonlinear Elastostatics and Nonlinear Bifurcation Theory.

SELECTED RECENT PAPERS

- (1) "Boundary Layer Effects for Inflated Elastic Rings and Shells," with M.C. Calderer, in preparation.
- (2) "Beck's Problem for Nonlinearly Viscoelastic Rods," with M.S. Chen, in preparation.
- (3) "Nonlinear Problems of Elasticity, Lectures Notes," in preparation, 400 pp.
- (4) "The Paradoxical Asymptotic Status of Massless Springs," in preparation.
- (5) "The Singular Global Buckling Problem for Anisotropic Plates," with P. Negron, in preparation.
- (6) "Travelling Waves in Nonlinearly Viscoelastic Media and Shock Structure in Elastic Media," with R. Malek-Madani, in preparation.
- (7) "Steady-State Problems of nonlinear Electro-Magneto-Thermo-Elasticity,
- (8) "Qualitative Properties of Large Buckled States of Spherical Shells," with K.G. Shih, University of Maryland TR85-30, 61 pp. To appear in Archive for Rational Mechanics and Analysis.
- (9) "Waves in Nonlinearly Viscoelastic Media, in Nonstrictly Hyperbolic Laws," with R. Malek-Madani, Contemporary Mathematics Series, AMS, edited by B.L. Keyfitz & H.C. Kranzer, to appear.
- (10) "The Drawing and Whirling of Strings: Singular Global Multiparameter Bifurcation Problems," with M. Reeken, University of Maryland Technical Report 1985, 58 pp. SIAM Journal of Mathematical Analysis, to appear.
- (11) "Bifurcation Problems Associated with Nonlinear Wave Propagation," in Proceedings of the Eighth Dundee Conference on Differential Equations, edited by B.D. Sleeman and R.J. Jarvis, Springer Lecture Notes in Mathematics, to appear.
- (12) "Asymptotic Shapes of Inflated Spheroidal Elastic Shells," with M.C. Calderer, Mathematical Proceedings of the Cambridge Philosophical Society, 97 (1985), 541-549.
- (13) "Asymptotic Shapes of Inflated Noncircular Elastic Rings" with M.C. Calderer, Mathematical Proceedings of the Cambridge Philosophical Society, 97 (1985), 357-379.

JOHN S. BARAS

Professor of Electrical Engineering

Director, Systems Research Center

John S. Baras was born in Piraeus, Greece, on March 13, 1948. He received a B.S. in Electrical Engineering with highest distinction from the National Technical University of Athens, Greece, in 1970. He received the M.S. and Ph.D. degrees in Applied Mathematics from Harvard University, Cambridge, Massachusetts in 1971 and 1973 respectively.

From 1973 to 1977, 1977 to 1982, he was an Assistant, Associate Professor and since 1982 a Professor of Electrical Engineering and member of the Applied Mathematics Faculty, University of Maryland. Since 1985 he has been Director of the Systems Research Center. He has held visiting research scholar positions with Stanford, MIT, Harvard and the University of California, Berkeley. Among his awards are: a 1978 Naval Research Laboratory Research Publication Award, the 1980 Outstanding Paper Award of the IEEE Control Systems Society and a 1983 Alan Berman Research Publication Award from NRL. He is a member of the IEEE Engineering R & D Committee. Professor Baras is a Fellow of IEEE.

RESEARCH INTERESTS

Professor Baras' main research interests are in stochastic systems, real-time parallel architectures for nonlinear signal processing, expert systems for control systems synthesis, distributed systems, queueing systems and scheduling.

SELECTED RECENT PAPERS

- (1) "Complex Variable Methods in Modelling and Control of Linear Distributed Parameter Systems." invited paper, to appear in *Frontiers in Applied Mathematics*.
- (2) "Decomposition and Decentralized Control System Design: A Review of Frequency Domain Methods," with W.H. Bennett, *Proc. 24th IEEE CDC*, 1985, pp. 1828-1835.
- (3) "A New Geometric Stability Margin for Feedback Systems," with W.H. Bennett, *Proc. 24th IEEE CDC*, 1985, pp.510-526.
- (4) "An Impulse Control Problem for a Stochastic PDE Arising in Nonlinear Filtering," with A. Bensoussan, *Proc. 24th IEEE CDC*, 1985, pp.1185-1187.
- (5) "Optimal Sensor Scheduling in Nonlinear Filtering of Diffusion Processes," with a. Bensoussan, submitted to *SIAM J. on Control and Optimization*.
- (6) "Expert Systems and VLSI Architectures in Real-Time Nongaussian Signal Processing," with A. LaVigna, *Proc. of 3rd Bad Honnef Conference*, 1985, to appear.
- (7) "Optimization Based, Direct, Frequency Domain Design of Controllers for Linear Distributed Systems," *Proc. 1985 IFAC Workshop on Control Applications of Nonlinear Programming and Optimization*, Pergamon Press, 1986.
- (8) "Two Competing Queues with Linear Costs and Exponential Service Requirements: The μ -rule is Often Optimal," with A.J.Dorsey and A. Makowski, *Advances in Applied Probability*, Vol. 16, No. 1, (1984) pg.8.
- (9) "K Competing Queues with Geometric Service Requirements and Linear Costs: The μ -rule is Always Optimal," with D.J. Ma and A. Makowski, *Systems and Control Letters*, Vol. 6, No. 3, (1985), pg. 173.

CARLOS A. BERENSTEIN

Professor of Mathematics

Carlos A. Berenstein was born in Buenos Aires, Argentina on August 7, 1944. He received a degree in mathematics (Licenciado) at the Universidad Nacional de Buenos Aires in 1966, an M.S. (1969) and the Ph.D. (1970) from the Courant Institute of Mathematical Sciences (New York University). In 1970-73 he was Benjamin Pierce Assistant Professor of Mathematics at Harvard University, since that time he has been at the University of Maryland where he became Associate Professor in 1976 and Full Professor in 1980. He has been a Visiting Professor at several institutions: Brandeis University, University of Paris, University of Bordeaux, Scuola Normale Superiore (Pisa), etc.

Dr. Berenstein is a member of the American Mathematical Society. He is currently the organizer of a Special Year in Complex Analysis at the University of Maryland, funded by the National Research Foundation and the Argonne University Association.

RESEARCH INTERESTS

Dr. Berenstein's interests lie in Complex Analysis, Harmonic Analysis and their applications to Engineering. These applications are based on recent developments in the theory of functions of several complex variables. They lie in image processing, signal processing, frequency domain methods for the control of distributed parameter systems, etc. Some of his students are currently working on the implementation of algorithms based on his work using MACSYMA, DELIGHT and MARYLIN.

SELECTED RECENT PAPERS

- (1) "An inverse Neumann problem," with P. Yang, to appear (1986). 33p.
- (2) "A local version of the two-circles theorem," with R. Gay, to appear in the Israel J. Math, 37p. "1-Inverses for polynomial matrices of non-constant rank," with D. Struppa, Systems and Control Letters, Vol. 6 (1986), pp.309-314.
- (3) "Solutions of convolution equations in convex sets," with D. Struppa, to appear in Amer. J. Math., 37p.
- (4) "Spectral synthesis on symmetric spaces," to appear in Contemporary Mathematics.
- (5) "Sur la synthese spectrale dans les espaces symmetriques," with R. Gay, to appear in J. Anal. Math.
- (6) "Deconvolution methods for multi-sensors," with P.S. Krishnaprasad and B.A. Taylor, (1984) 67p.
- (7) "Ideals generated by exponential-polynomials," with A. Yger, to appear in Advances in Mathematics, 132 p.
- (8) "On the equilibria of rigid spacecraft with rotors," with P.S. Krishnaprasad, Systems and Control Letters, Vol. 4 (1984), pp.157-163.
- (9) "Le probleme de la deconvolution," with A. Yger, J. Funct. Anal., Vol. 54 (1983), pp.113-160.

JOHN HENRY MADDOCKS

Assistant Professor of Mathematics

John Maddocks was born in the United Kingdom in 1958. He received his B.S. in Mathematics with honors in 1978 from the University of Glasgow. In 1982 he received his Ph.D. from Balliol College, University of Oxford.

Before becoming an Assistant Professor of Mathematics at the University of Maryland in 1984, Dr. Maddocks was an Assistant Professor at Stanford University (1981-82), a Junior Research Fellow at the Queen's College, University of Oxford (1983-84) and a Senior Fellow at the Institute for Mathematics and Its Application, University of Minnesota (1984-85). He is a member of AMS, SIAM and Society for Natural Philosophy. Additionally, Dr. Simo participates as a reviewer for N.S.F., International J. of Solids and Structures and J. of Applied Math. and Optimization.

RESEARCH INTERESTS

Dr. Simo's research interests are in the area of Elastostatics and Nonlinear Bifurcation Theory.

SELECTED RECENT PAPERS

- (1) "Stability of nonlinearly elastic rods," Arch. Rat. Mech. Anal., 85, 1984, pp. 311-354.
- (2) "Dynamical theory of the elastica," with R.E. Caflisch, Proc. Roy. Soc. Edin. A., Vol. 99, pp. 1-23, 1984.
- (3) "Restricted quadratic forms and their application to bifurcation and stability in constrained variational principles," SIAM J. Math. Anal., Vol. 16 1985, pp. 47-68.
- (4) "A model for twinning," with G.P. Parry, J. Elasticity, in press. IMA Preprint Series No. 125, 1985.
- (5) "Stability and folds," IMA Preprint Series No. 174, 1985. Submitted to Arch. Rat. Mech. Anal.
- (6) "Mechanics of ropes," with J.B. Keller, IMA Preprint Series No. 183, 1985. Submitted to SIAM J. Appl. Math.
- (7) "Stability of the elastica," Proc. of Nato/London Math. Soc. Advanced Study Institute. Ed. J.M. Ball, Reidel, 1983.

JERROLD E. MARSDEN

Professor of Mathematics
University of California, Berkeley

Jerrold Marsden was born in British Columbia, Canada August 17, 1942. He received his B.S. from the University of Toronto in 1965 and his Ph.D from Princeton University in 1967.

Dr. Marsden has served as an Instructor at Princeton University (1967-68); Lecturer in Mathematics at Berkeley (1968-69); Assistant Professor, University of Toronto (1970-71); Assistant Professor, University of California, Berkeley (1971-72); and Associate Professor, University of California, Berkeley (1972-76). Since 1976 he has been a Professor of Mathematics at Berkeley. Additionally, Dr. Marsden has held many visiting research, professor, fellow and scholar titles nationally and internationally. Dr. Marsden has presented numerous invited talks and conference lectures and received a number of awards for essays and research. He serves the scientific community as an editor or reviewer for many scholarly publications. Dr. Marsden has written 14 books and over 100 articles on various topics in mathematics.

RESEARCH INTERESTS

Dr. Marsden's research interests are in the areas of Nonlinear Dynamics, Hamiltonian Systems, Chaotic Dynamics, Dynamical Systems and Stability among others.

SELECTED RECENT PAPERS

- (1) "Constraints and momentum maps for relativistic classical fields," with Gotay, Isenberg, Montgomery, Sniatycki and Yasskin, in preparation.
- (2) "Bifurcation to quasi-periodic Tori in the interaction of steady state and Hopf bifurcations," with J. Scheurle, accepted, *SIAM J. Math. Ann.*
- (3) "On the rotated stress tensor and the material version of the Doyle-Ericksen formula," with J.C. Simo, accepted, *Arch. Rat. Mech. Ann.*
- (4) "The York Map is a canonical transformation," with J. Isenberg, accepted, *J. of Geometry and Physics.*
- (5) "Temporal and spatial chaos in a van der Waals fluid due to periodic thermal fluctuations," with M. Slemrod, accepted, *Adv. in Appl. Math.*
- (6) "Averaging, horseshoes, and exponentially small Melnikov functions," with P. Holmes and J. Scheurle, in preparation.
- (7) "Quasiconvexity, second variations and the energy criterion in nonlinear elasticity," with J. Ball, accepted, *Arch. Rat. Mech. Ann.*
- (8) "Nonlinear stability of fluid and plasma systems," with D. Holm, T. Ratiu and A. Weinstein, accepted, *Physics Reports.*
- (9) "The Hamiltonian structure of general relativistic perfect fluids," with Bao and Walton, preprint.
- (10) "Generalized Poisson brackets and nonlinear Liapunov stability - application to reduced MHD" with Hazeltine, Holm and Morrison, accepted, *Proc. Plasma Phys. Conference, Lausanne, June 1984.*
- (11) "Chaos in dynamical systems by the Poincare-Melnikov-Arnold method" *Proc. ARO Workshop, March 1984.*

JUAN C. SIMO

Assistant Professor, Mechanical Engineering
Stanford University, California

Juan Simo was born in Spain. He received his degree of E.T.S: Ing. Caminos Canales y Puertos in 1976 from the Universidad Politecnica, Madrid, Spain. In 1978, Dr. Simo received his M.S. degree in Metodos Cuantitativos from Escuela de Organizacion Industrial, Madrid, Spain. The M.S. and Ph.D. were received from the University of California at Berkeley in Civil Engineering in 1980 and 1982 respectively.

Dr. Simo has held positions as Research Assistant, Lecturer, Post-Doctoral Fellow and Assistant Research Engineer at the University of California at Berkeley (1980-1985). Most recently he has been at Stanford University as an Instructor (1984), Lecturer (1985) and as an Assistant Professor (1985-present). He has been invited to present his work at a number of conferences, symposium and lectures in the United States as well as internationally.

RESEARCH INTERESTS

Dr. Simo's research interests are in the areas of Nonlinear Theory and Elastostatics.

SELECTED RECENT PAPERS

- (1) "Remarks on the Patch Test and Stummel's Counterexample," with R.L. Taylor, International Journal of Numerical Methods in Engineering, to be submitted.
- (2) "On a fully Three-dimensional finite strain viscoelastic damage model: Formulation and computational aspects," Comp. Meth. Appl. Mech. Engng., submitted for publication.
- (3) "On the Dynamics of Flexible Beams Subject to Large Overall Motions— The Plane Case," with L. Vu-Quoc, Electronics Research Laboratory Memorandum No. UCB/ERL M85/63, University of California, Berkeley, submitted for publication in J. Applied Mechanics.
- (4) "A consistent return mapping algorithm for plane strain elastoplasticity," with R.L. Taylor, Int. J. Num. Methd. Engng., (in press).
- (5) "A 3-Dimensional Finite Strain Rod Model: Geometric and Computational Aspects," with L.V. Quoc, Comp. Mth. Appl. Mech. Engng., in press.
- (6) "On the Variational Structure of Assumed Strain Methods," with T.J.R. Hughes, J. Appl. Mechanics, in press.
- (7) "An Analysis of a New Class of Integration Algorithms for Elastoplastic Constitutive Relations," with M. Ortiz, Int. J. Num. Meth. Engng., in press.
- (8) "A 3-Dimensional Finite Deformation Viscoelastic Model Accounting for Damage Effects," with R.L. Taylor, Report No. UCB/SESM-85/02, Department of Civil Engineering, University of California, Berkeley, 1985.
- (9) "Bending and Membrane Elements for the Analysis of Thick and Thin Shells,
- (10) "Augmented Lagrangian Formulations for the Finite Element Solution of Contact Problems," with P. Wriggers and R.L. Taylor, NUMETA Conference, Swansea, Wales, January 1985.
- (11) "Rate Constitutive Equations: Are they Necessary?," with M. Ortiz, K.S. Pister and R.L. Taylor, ASME Winter Annual Meeting, New Orleans 1984.

END

DATE

FILMED

FEB.

1988